

Quantum Field Theory

Set 8

Exercise 1: Noether's currents in classical mechanics

Consider the general Lagrangian of a classical system

$$L = \sum_a \frac{m_a}{2} (\dot{\vec{q}}_a)^2 - \sum_{a \neq b} V(|\vec{q}_a - \vec{q}_b|).$$

Using the Noether procedure compute the Noether current associated to the following transformations

- time translations:

$$t' = t - \alpha, \quad \vec{q}'_a(t') = \vec{q}_a(t)$$

- coordinate translations:

$$t' = t, \quad \vec{q}'_a(t) = \vec{q}_a(t) + \vec{\alpha}$$

- coordinate rotations:

$$t' = t, \quad q'^i_a(t) = \mathcal{R}^i_j q^j_a(t)$$

Exercise 2: Noether's current for $U(1)$ global symmetry

Consider the Lagrangian of a free *complex* massive scalar field $\phi(x) = \phi_1(x) + i\phi_2(x)$:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi.$$

Show that the Lagrangian density is invariant under the following transformation

$$x_\mu \longrightarrow x_\mu, \quad \phi(x) \longrightarrow e^{i\alpha} \phi(x),$$

where α does not depend on x .

- Compute the Noether current associated to this symmetry and show that is conserved only if one uses the equations of motion.
- How does the result change if we add to the Lagrangian an interaction term of the form $V(\phi\phi^\dagger)$?

Exercise 3: Noether's charge as generator of transformations

Given a Lagrangian density $\mathcal{L}(\phi_a, \partial_\mu \phi_b)$ consider the symmetry transformation defined by

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = f(x) \simeq x^\mu - \epsilon^\mu_i \alpha^i, \\ \phi_a(x) &\longrightarrow \phi'_a(x') = D[\phi]_a(f^{-1}(x')) \simeq \phi_a(x') + \alpha^i \Delta_{ai}(\phi(x')), \end{aligned}$$

- Show that charge Q_i built starting from the Noether's current is the generator of the transformation:

$$\delta_\alpha \phi_a(x) \equiv \phi'_a(x) - \phi_a(x) = \alpha^i \{Q_i, \phi_a(x)\} = \alpha^i \Delta_{ai}(x).$$

- Compute explicitly the charges associated to spacetime translation and Lorentz transformations in a scalar field theory.
- Using the formalism of the Poisson brackets check that the charges related to translations and spatial rotations generate the respective infinitesimal transformations.

Exercise 4: $SU(2)$ Noether's current

Consider a pair of complex scalar fields $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ transforming according to the representation $j = 1/2$ of a global $SU(2)$ symmetry (do not confuse this symmetry with the Lorentz transformations: they are completely uncorrelated):

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = x^\mu, \\ \Phi(x) &\longrightarrow \Phi'(x') = \mathcal{U}\Phi(x), \\ \phi_a(x) &\longrightarrow \phi'_a(x') = \mathcal{U}_a{}^b \phi_b(x), \quad a, b = 1, 2. \end{aligned}$$

where $\mathcal{U}$ is a $SU(2)$ matrix. Given the Lagrangian density

$$\mathcal{L} = \partial^\mu \Phi^\dagger \partial_\mu \Phi - m^2 \Phi^\dagger \Phi - \frac{\lambda}{2} (\Phi^\dagger \Phi)^2,$$

- Show that the Lagrangian density is invariant under $SU(2)$ transformations.
- Compute the Noether current for this symmetry.
- Add to the above Lagrangian the interaction with a real *triplet* $\vec{A} = (A_1, A_2, A_3)$ of $SU(2)$

$$\tilde{\mathcal{L}} = \mathcal{L} + \frac{1}{2} \partial^\mu A^T \partial_\mu A + A_i \Phi^\dagger \sigma^i \Phi.$$

Is the new Lagrangian density invariant under $SU(2)$ transformations involving the field Φ and $\vec{A}$?

- If yes, what other terms with dimension less or equal than four can be added to the Lagrangian so that it remains invariant under $SU(2)$ (and the Lorentz group)?
Hint: recall the following properties of Pauli matrices

$$\sigma^i \sigma^j = \delta^{ij} \mathbf{1} + 2i\varepsilon^{ijk} \sigma^k, \quad \sigma_{\alpha\beta}^i \sigma_{\gamma\delta}^i = 2\delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\beta} \delta_{\gamma\delta}.$$